Kinematics of Robot Manipulator

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Robotics

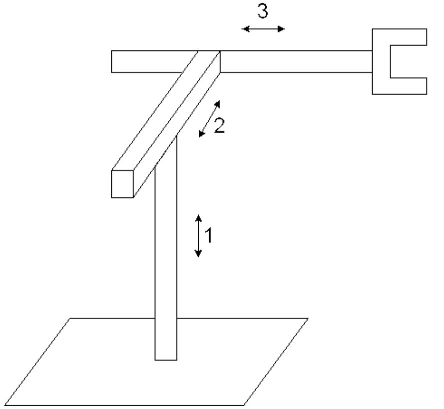
Manipulators

- Robot arms, industrial robot
  - Rigid bodies (links) connected by joints
  - Joints: revolute or prismatic
  - Drive: electric or hydraulic
  - End-effector (tool) mounted on a flange or plate secured to the wrist joint of robot

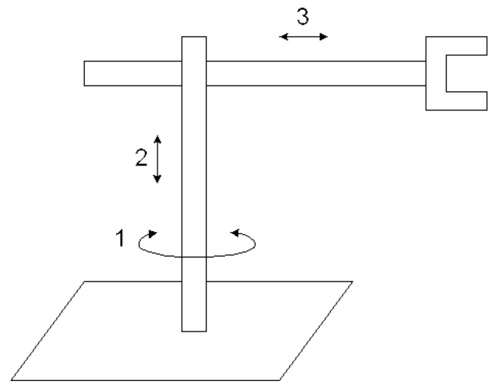
Robotics

Manipulators

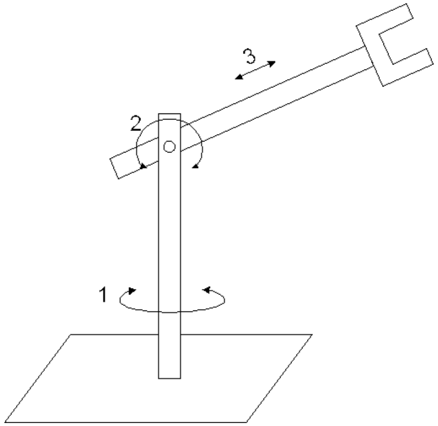
- Robot Configuration:



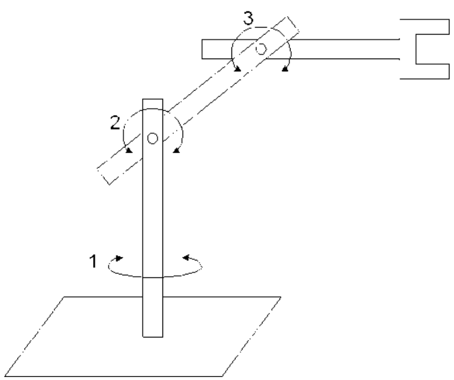
Cartesian: PPP



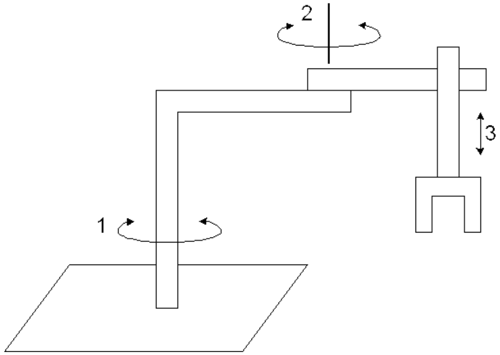
Cylindrical: RPP



Spherical: RRP

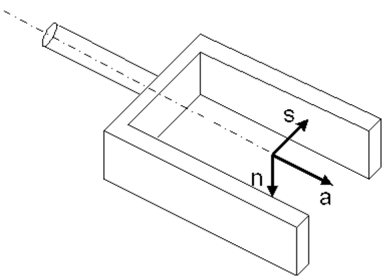


Articulated: RRR



SCARA: RRP

(Selective Compliance Assembly Robot Arm)



Hand coordinate:

n: normal vector; **s**: sliding vector;

a: approach vector, normal to the tool mounting plate

Robotics

Manipulators

- Robot Specifications
  - Number of Axes
    - Major axes, (1-3) => Position the wrist
    - Minor axes, (4-6) => Orient the tool
    - Redundant, (7-n) => reaching around obstacles, avoiding undesirable configuration
  - Degree of Freedom (DOF)
  - Workspace
  - Payload (load capacity)
  - Precision v.s. Repeatability

how accurately a specified point can be reached

how accurately the same position can be reached if the motion is repeated many times

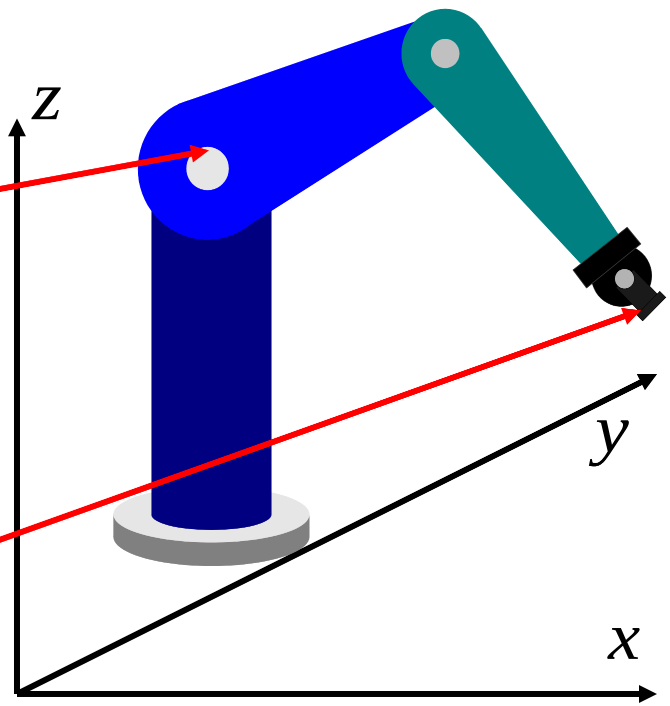
What is Kinematics

- Forward kinematics

Given joint variables

$$q = (q_1, q_2, q_3, q_4, q_5, q_6, \cdots q_n)$$

$$Y = (x, y, z, O, A, T)$$



End-effector position and orientation, -Formula?

Robotics

What is Kinematics

- Inverse kinematics

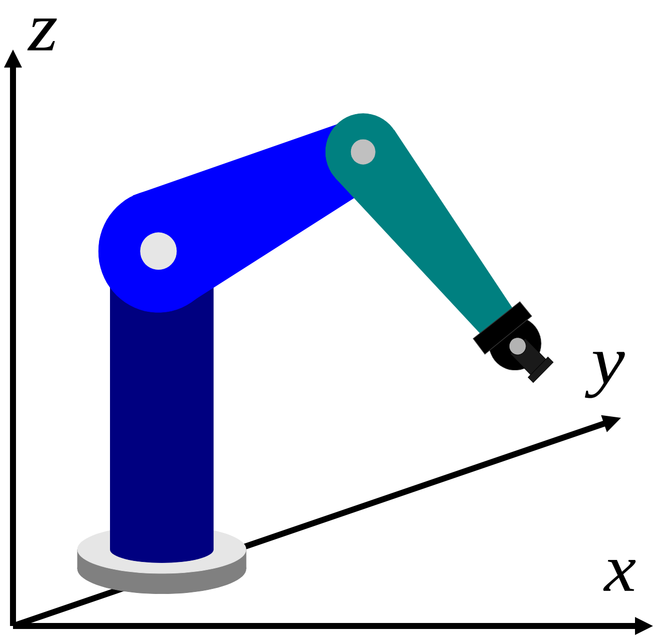
End effector position

and orientation

$$(x, y, z, O, A, T)$$

$$\Downarrow$$

$$q = (q_1, q_2, q_3, q_4, q_5, q_6, \cdots q_n)$$



Joint variables -Formula?

Robotics

Example 1

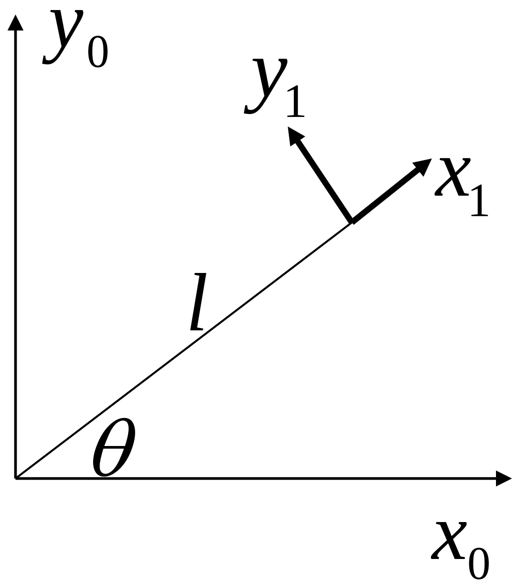
Forward kinematics

$$x_1 = l\cos\theta$$

$$y_1 = l\sin\theta$$

Inverse kinematics

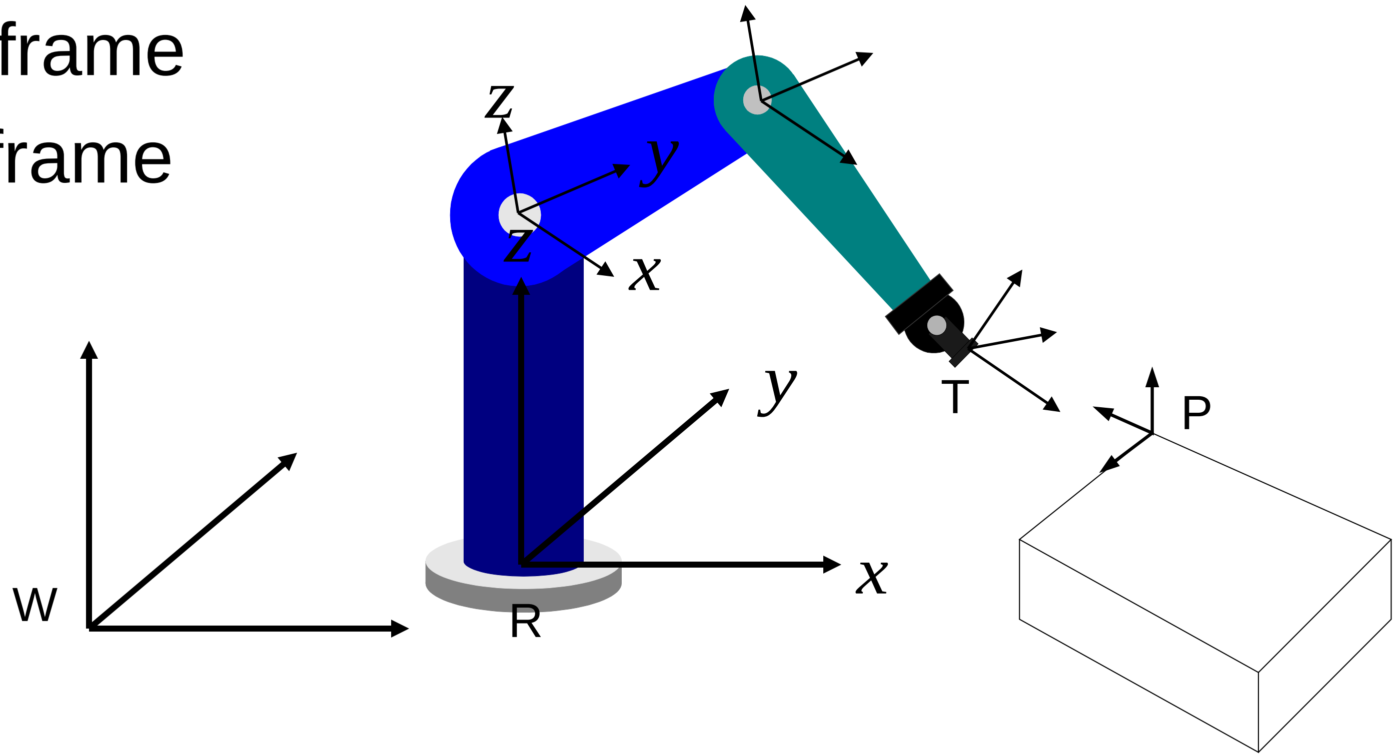
$$\theta = \cos^{-1}(x_1 / l)$$



Robotics

Preliminary

- Robot Reference Frames
  - World frame
  - Joint frame
  - Tool frame



Robotics

Preliminary

- Coordinate Transformation
  - Reference coordinate frame OXYZ
  - Body-attached frame O' uvw

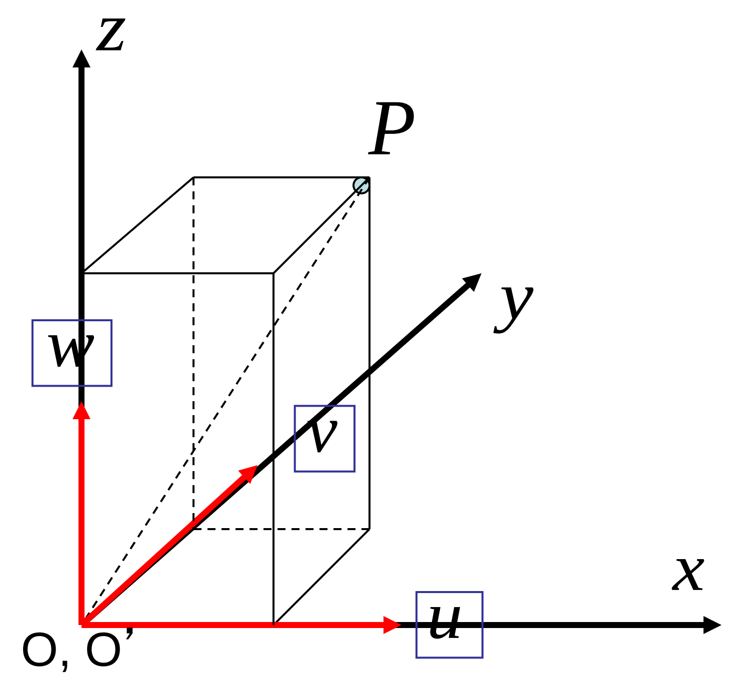
Point represented in OXYZ:

$$P_{xyz} = [p_x, p_y, p_z]^T$$

$$\vec{P}_{xyz} = p_x \mathrm{i}_\mathrm{x} + p_y \mathrm{j}_\mathrm{y} + p_z \mathrm{k}_\mathrm{z}$$

Point represented in O' uvw:

$$\vec{P}_{uvw} = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$$



Two frames coincide ==> $p_u = p_x \quad p_v = p_y \quad p_w = p_z$

Robotics

Preliminary

Properties: Dot Product

Let x and y be arbitrary vectors in R^3 and θ be the angle from x to y, then

$$x \cdot y = |x||y| \cos\theta$$

Properties of orthonormal coordinate frame

- Mutually perpendicular

$$\vec{i} \cdot \vec{j} = 0$$

$$\vec{i} \cdot \vec{k} = 0$$

$$\vec{k} \cdot \vec{j} = 0$$

- Unit vectors

$$|\vec{i}| = 1$$

$$|\vec{j}| = 1$$

$$|\vec{k}| = 1$$

Robotics

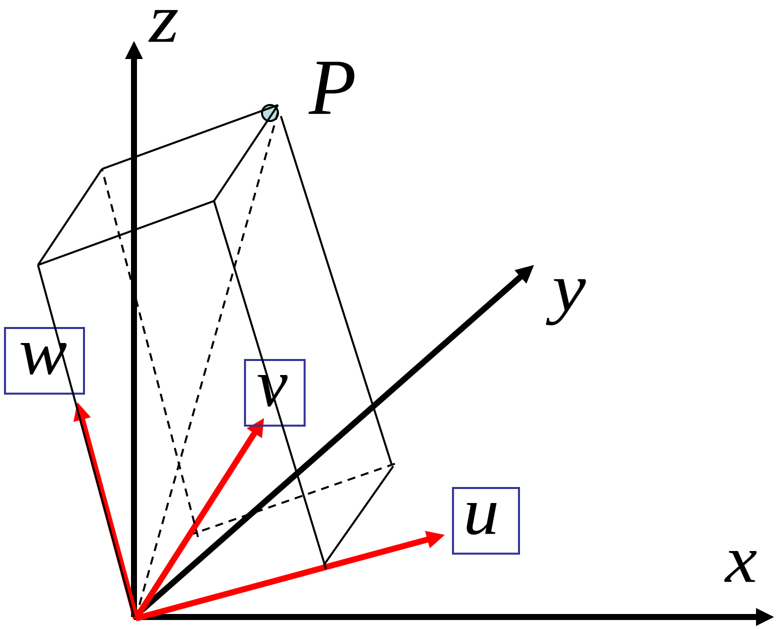
Preliminary

- Coordinate Transformation
  - Rotation only

$$\vec{P}_{xyz} = p_x \mathrm{i}_\mathrm{x} + p_y \mathrm{j}_\mathrm{y} + p_z \mathrm{k}_\mathrm{z}$$

$$\vec{P}_{uvw} = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$$

$$P_{xyz} = RP_{uvw}$$



How to relate the coordinate in these two frames?

Robotics

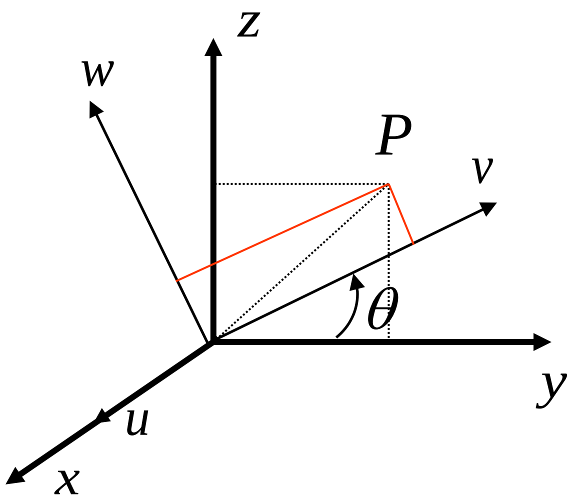
Preliminary

- Basic Rotation Matrix

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathrm{i_x} \cdot \mathrm{i_u} & \mathrm{i_x} \cdot \mathrm{j_v} & \mathrm{i_x} \cdot \mathrm{k_w} \\ \mathrm{j_y} \cdot \mathrm{i_u} & \mathrm{j_y} \cdot \mathrm{j_v} & \mathrm{j_y} \cdot \mathrm{k_w} \\ \mathrm{k_z} \cdot \mathrm{i_u} & \mathrm{k_z} \cdot \mathrm{j_v} & \mathrm{k_z} \cdot \mathrm{k_w} \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

  – Rotation about x-axis with θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

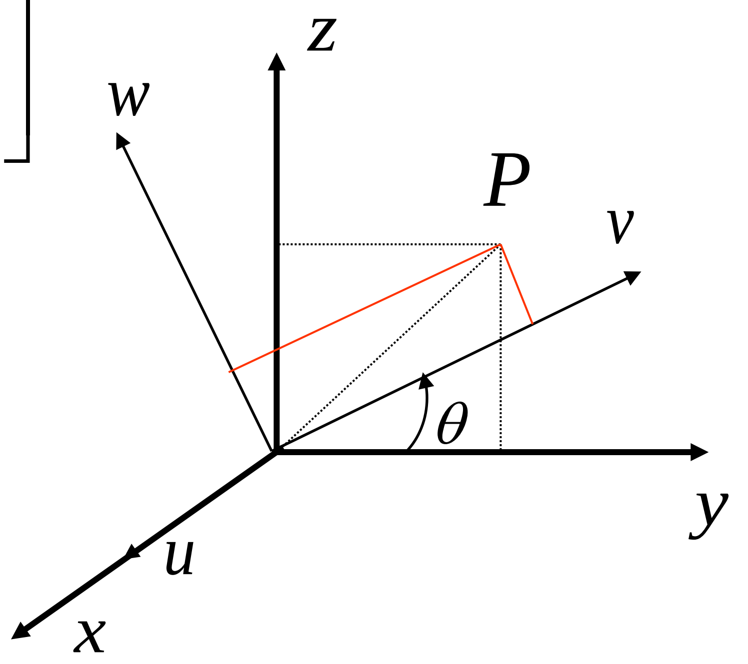


Robotics

Preliminary

- Is it True?
  - Rotation about x axis with θ

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$



$$p_x = p_u$$

$$p_y = p_v \cos\theta - p_w \sin\theta$$

$$p_z = p_v \sin\theta + p_w \cos\theta$$

Robotics

Basic Rotation Matrices

– Rotation about x-axis with θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

– Rotation about y-axis with θ

$$Rot(y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix}$$

– Rotation about z-axis with θ

$$Rot(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{xyz} = RP_{uvw}$$

Robotics

Preliminary

- Basic Rotation Matrix

$$R = \begin{bmatrix} \mathrm{i_x} \cdot \mathrm{i_u} & \mathrm{i_x} \cdot \mathrm{j_v} & \mathrm{i_x} \cdot \mathrm{k_w} \\ \mathrm{j_y} \cdot \mathrm{i_u} & \mathrm{j_y} \cdot \mathrm{j_v} & \mathrm{j_y} \cdot \mathrm{k_w} \\ \mathrm{k_z} \cdot \mathrm{i_u} & \mathrm{k_z} \cdot \mathrm{j_v} & \mathrm{k_z} \cdot \mathrm{k_w} \end{bmatrix}$$

$$P_{xyz} = RP_{uvw}$$

  – Obtain the coordinate of P_{uvw} from the coordinate of P_{xyz}

Dot products are commutative!

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \mathrm{i_u} \cdot \mathrm{i_x} & \mathrm{i_u} \cdot \mathrm{j_y} & \mathrm{i_u} \cdot \mathrm{k_z} \\ \mathrm{j_v} \cdot \mathrm{i_x} & \mathrm{j_v} \cdot \mathrm{j_y} & \mathrm{j_v} \cdot \mathrm{k_z} \\ \mathrm{k_w} \cdot \mathrm{i_x} & \mathrm{k_w} \cdot \mathrm{j_y} & \mathrm{k_w} \cdot \mathrm{k_z} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$P_{uvw} = QP_{xyz}$$

$$Q = R^{-1} = R^T$$

$$QR = R^T R = R^{-1} R = I_3$$

<== 3X3 identity matrix

Robotics

Example 2

- A point $a_{uvw} = (4,3,2)$ is attached to a rotating frame, the frame rotates 60 degree about the OZ axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation.

$$a_{xyz} = Rot(z,60)a_{uvw}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

Example 3

- A point $a_{xyz} = (4,3,2)$ is the coordinate w.r.t. the reference coordinate system, find the corresponding point a_{uvw} w.r.t. the rotated OU-V-W coordinate system if it has been rotated 60 degree about OZ axis.

$$a_{uvw} = Rot(z,60)^T a_{xyz}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.598 \\ -1.964 \\ 2 \end{bmatrix}$$

Composite Rotation Matrix

- A sequence of finite rotations
  - matrix multiplications do not commute
  - rules:
    - if rotating coordinate O-U-V-W is rotating about principal axis of OXYZ frame, then ***Pre-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix
    - if rotating coordinate OUVW is rotating about its own principal axes, then ***post-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix

Example 4

- Find the rotation matrix for the following operations:

Rotation ϕ about OY axis

Rotation θ about OW axis

Rotation α about OU axis

Answer...

$$R = Rot(y,\phi) I_3 Rot(w,\theta) Rot(u,\alpha)$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

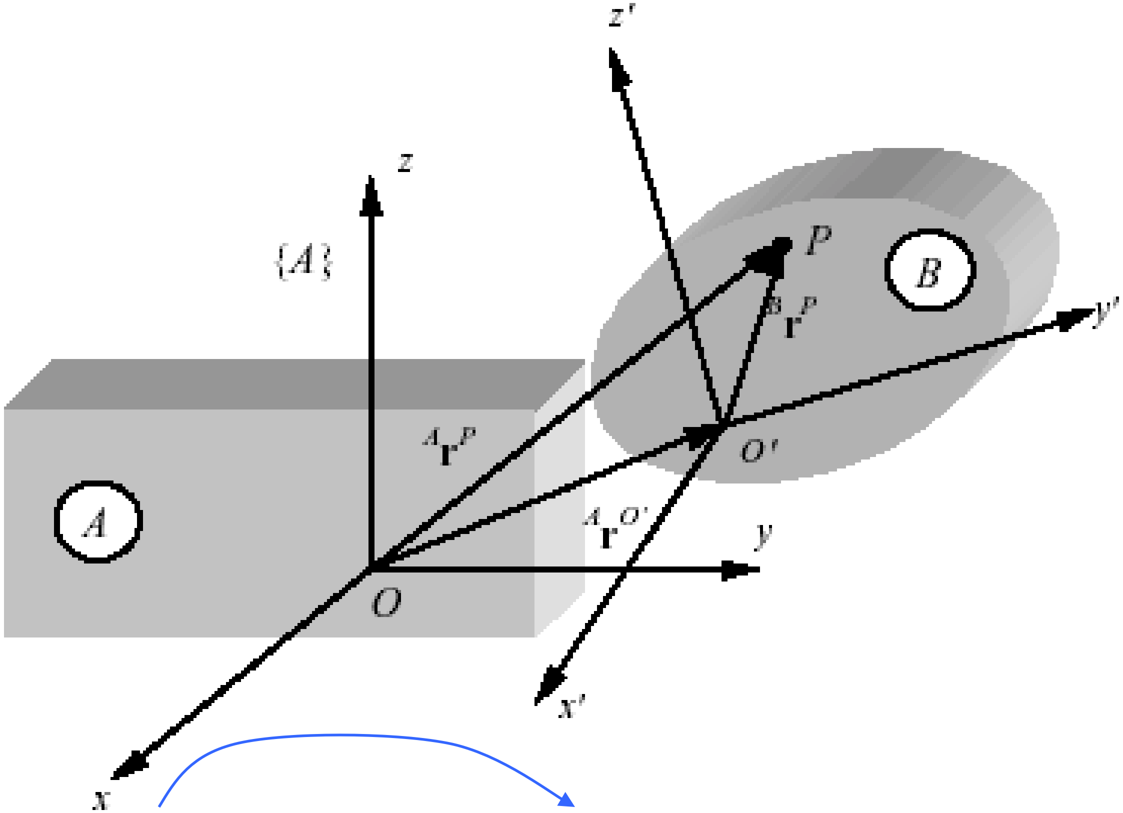
Pre-multiply if rotate about the OXYZ axes
Post-multiply if rotate about the OUVW axes

Robotics

Coordinate Transformations

- position vector of P in $\{B\}$ is transformed to position vector of P in $\{A\}$

- description of $\{B\}$ as seen from an observer in $\{A\}$



Rotation of $\{B\}$ with respect to $\{A\}$

$${}^{A}\mathbf{r}^{P} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{r}^{P} + {}^{A}\mathbf{r}^{O'}$$

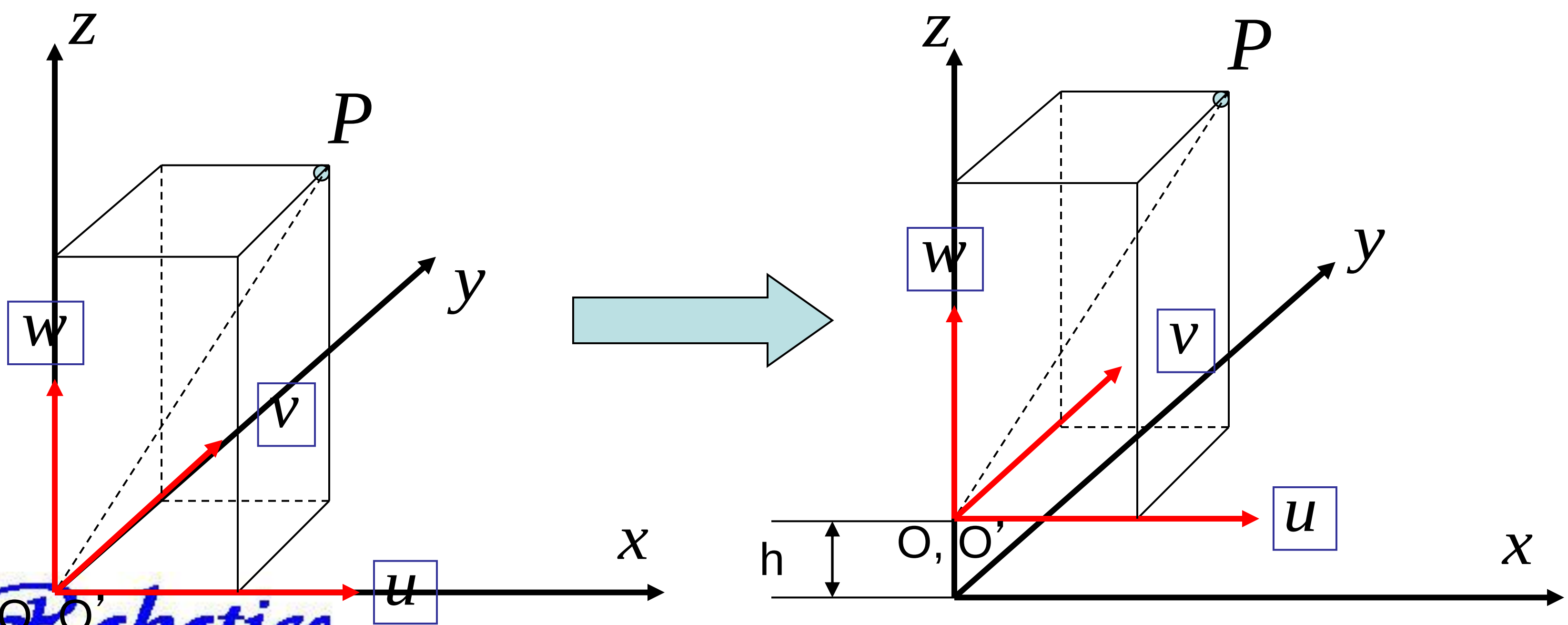
Translation of the origin of $\{B\}$ with respect to origin of $\{A\}$

Robotics

Example 5

- Translation along Z-axis with h:

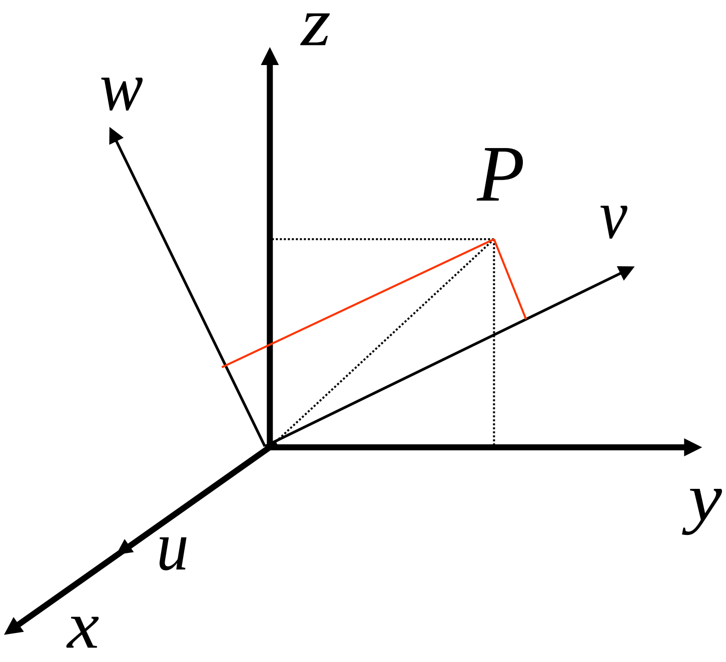
$$Trans(z,h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix} = \begin{bmatrix} p_u \\ p_v \\ p_w + h \\ 1 \end{bmatrix}$$



Example 6

- Rotation about the X-axis by

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}p_u\\ p_v\\ p_w\\ 1\end{bmatrix}$$



Robotics

Homogeneous Transformation

- Composite Homogeneous Transformation Matrix
- Rules:
  - Transformation (rotation/translation) w.r.t (X,Y,Z) (OLD FRAME), using pre-multiplication
  - Transformation (rotation/translation) w.r.t (U,V,W) (NEW FRAME), using post-multiplication

Robotics

Example 7

- Find the homogeneous transformation matrix (T) for the following operations:

Rotation α about OX axis

Translation of a along OX axis

Translation of d along OZ axis

Rotation of θ about OZ axis

$$T = T_{z,\theta} T_{z,d} T_{x,a} T_{x,\alpha} I_{4\times4}$$

$Answer:$

$$= \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

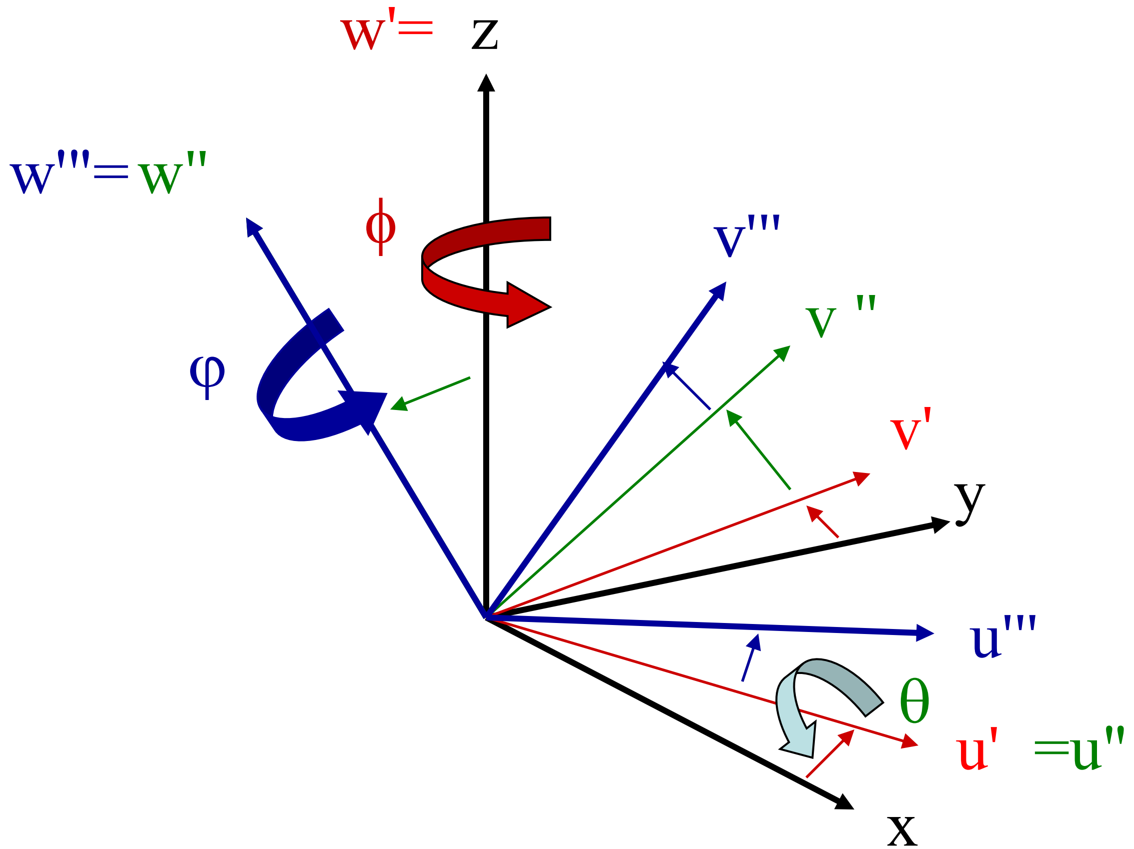
Orientation Representation

- Euler Angles Representation (ϕ, θ, ψ)
  - Many different types
  - Description of Euler angle representations

	Euler Angle I	Euler Angle II	Roll-Pitch-Yaw
Sequence	ϕ about OZ axis	ϕ about OZ axis	ψ about OX axis
of	θ about OU axis	θ about OV axis	θ about OY axis
Rotations	ψ about OW axis	ψ about OW axis	ϕ about OZ axis

Robotics

Euler Angle I, Animated



Robotics

Orientation Representation

- Euler Angle I

$$R_{z\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{u'\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$R_{w''\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Robotics

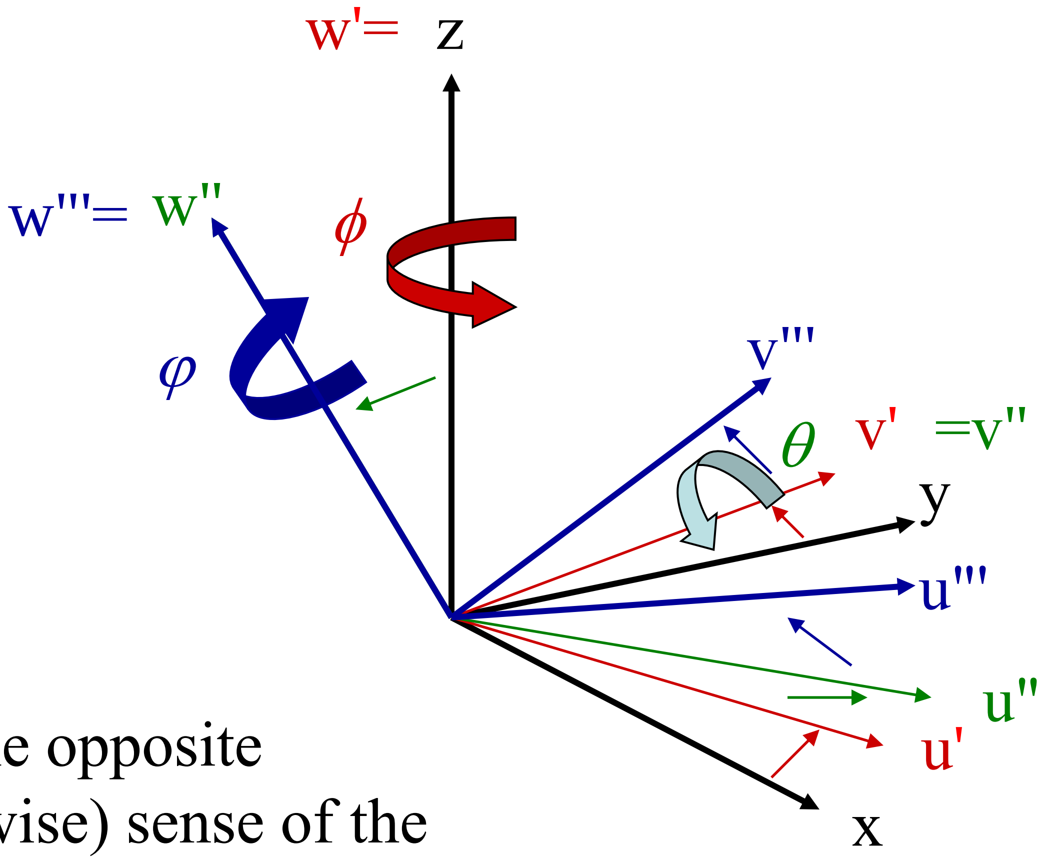
Euler Angle I

Resultant eulerian rotation matrix:

$$R = R_{z\phi} R_{u'\theta} R_{w''\varphi}$$

$$\begin{pmatrix} \cos\phi\cos\varphi - \sin\phi\sin\varphi\cos\theta & -\cos\phi\sin\varphi - \sin\phi\cos\varphi\cos\theta & \sin\varphi\sin\theta \\ \sin\phi\cos\varphi + \cos\phi\sin\varphi\cos\theta & -\sin\phi\sin\varphi + \cos\phi\cos\varphi\cos\theta & -\cos\phi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{pmatrix}$$

Euler Angle II, Animated



Note the opposite (clockwise) sense of the third rotation, ϕ.

Robotics

Orientation Representation

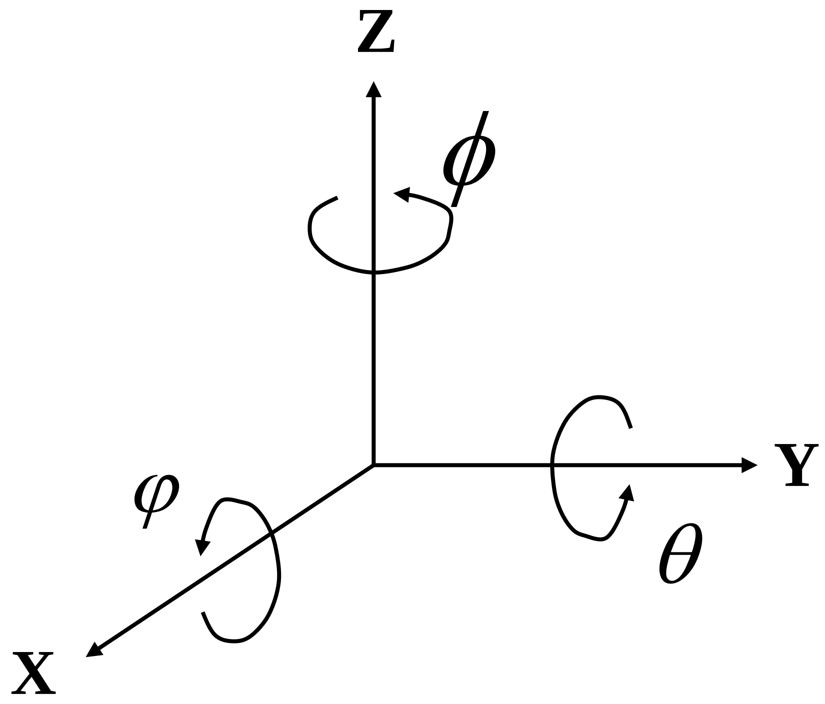
- Matrix with Euler Angle II

$$
\begin{pmatrix}
-\sin\phi\sin\varphi + \cos\phi\cos\varphi\cos\theta & -\sin\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \cos\phi\sin\theta \\
\cos\phi\sin\varphi + \sin\phi\cos\varphi\cos\theta & \cos\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \sin\varphi\sin\theta \\
-\cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta
\end{pmatrix}
$$

Quiz: How to get this matrix ?

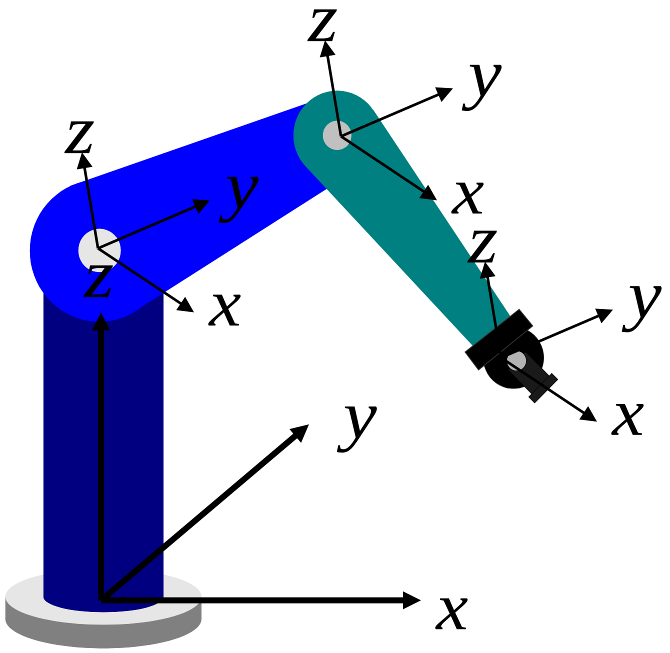
Orientation Representation

- Description of Roll Pitch Yaw



Quiz: How to get rotation matrix ?

Thank you!



Robotics