2.4 Definability of PL' and PL'' Languages

Definition: A *well-formed formula* (or *wff*) of PL' is defined recursively as follows:

(1) Any statement letter is a well-formed formula.
(2) If α is a well-formed formula, then so is ¬α.
(3) If α and β are well-formed formulas, then so is (α → β).
(4) Nothing that cannot be constructed by steps of (1)-(3) is a well-formed formula.

¬(α → ¬β) = (α&β).

α	B		¬	(α	→	¬	β)
T	T		T	T	F	F	T
T	F		F	T	T	T	F
F	T		F	F	T	F	T
F	F		F	F	T	T	F

(¬α → β) = (α v β).

α	β		(¬	α	→	β)
T	T		F	T	T	T
T	F		F	T	T	F
F	T		T	F	T	T
F	F		T	F	F	F

¬((α → β) → ¬(β → α)) = (α↔β).

α	β		¬	((α	→	β)	→	¬	(β	→	α))
T	T		T	T	T	T	F	F	T	T	T
T	F		F	T	F	F	T	F	F	T	T
F	T		F	F	T	T	T	T	T	F	F
F	F		T	F	T	F	F	F	F	T	F

It is possible to define PL using signs '¬' and 'v'.

- (α &β) would be defined as ¬(¬α v ¬β),
- (α →β) would be defined as (¬α v β),
- (α ↔ β) would be defined as ¬(¬(¬α v β) v ¬(¬β v α)).

Similarly, we could instead use '¬' and '&'.

- (αvβ) would be defined as ¬(¬α & ¬β),
- (α→β) would be defined as ¬(α & ¬β),
- (α ↔ β) would be defined as (¬(α & ¬β) & ¬(β & ¬α)).

Definition: A *well-formed formula* (or *wff*) of PL" is defined recursively as follows:

(1) Any statement letter is a well-formed formula.
(2) If α and β are well-formed formulas, then so is (α | β).
(3) Nothing that cannot be constructed by steps of (1)-(2) is a well-formed formula.

Sheffer stroke (NAND)

α	β	(α | β)
T	T	F
T	F	T
F	T	T
F	F	T

((α|β)|(α|β)) = (α&β)

α	β		((α	|	β)	|	(α	|	β))
T	T		T	F	T	T	T	F	T
T	F		T	T	F	F	T	T	F
F	T		F	T	T	F	F	T	T
F	F		F	T	F	F	F	T	F

((α|α) | (β|β)) = (αvβ)

α	β		((α	|	A)	|	(β	|	β))
T	T		T	F	T	T	T	F	T
T	F		T	F	T	T	F	T	F
F	T		F	T	F	T	T	F	T
F	F		F	T	F	F	F	T	F

(α|α) = ¬α:

α		(α	|	α)
T		T	F	T
F		F	T	F

(α | (β | β)) is equivalent to (α→β):

α	β		(α	|	(β	|	β))
T	T		T	T	T	F	T
T	F		T	F	F	T	F
F	T		F	T	T	F	T
F	F		F	T	F	T	F

(((α | α) | (β | β)) | (α | β)) to (α ↔ β):

α	β		(((α	|	α)	|	(β	|	β))	|	(α	|	β))
T	T		T	F	T	T	T	F	T	T	T	F	T
T	F		T	F	T	T	F	T	F	F	T	T	F
F	T		F	T	F	T	T	F	T	F	F	T	T
F	F		F	T	F	F	F	T	F	T	F	T	F

Another way, PL" is defined using '↓', *Sheffer dagger*(NOR).

- ¬α is defined as (α ↓ α);
- (α v β) is defined as ((α ↓ β) ↓ (α ↓ β));
- (α & β) is defined as ((α ↓ α) ↓ (β ↓ β));
- and so on for the other operators

α	β	(α ↓ β)
T	T	F
T	F	F
F	T	F
F	F	T

The advantage of a rich language like PL is that it conforms better with our ordinary reasoning.
The advantage of a sparse language such as PL' or PL" is that it simplifies the logical language.