Syntax and semantic of FOL

Syntax and Semantic of FOL

- Model of propositional logic
  - Sets of truth values for the propostional symbols
- Models of first-order logic
  - Unlike in propositional logic, models in FOL are more than just truth assignments to the objects… they are relationships among the objects
- Domain of model
  - The set of objects (domain element) it contains

First-order logic syntax

Sentence	$\rightarrow$	***AtomicSentence***
	$\mid$	**(*Sentence Connective Sentence*)**
	$\mid$	***Quantifier Variable,... Sentence***
	$\mid$	$\neg$ ***Sentence***
AtomicSentence	$\rightarrow$	*Predicate*(*Term*,...) $\mid$ *Term* = *Term*
Term	$\rightarrow$	*Function*(*Term*,...)
	$\mid$	*Constant*
	$\mid$	*Variable*
Connective	$\rightarrow$	$\Rightarrow \mid \wedge \mid \vee \mid \Leftrightarrow$
Quantifier	$\rightarrow$	$\forall \mid \exists$
Constant	$\rightarrow$	$A \mid X_1 \mid John \mid \cdots$
Variable	$\rightarrow$	$a \mid x \mid s \mid \cdots$
Predicate	$\rightarrow$	$Before \mid HasColor \mid Raining \mid \cdots$
Function	$\rightarrow$	$Mother \mid LeftLeg \mid \cdots$

First-order logic syntax

- Symbol:
- **Constant Symbols**
  - A, B, Bob, Alice, Hat
- **Predicate Symbols**
  - is, onHead, hasColor, person
- **Function Symbols**
  - Mother, leftLeg

- Each predicate and function symbol has an **arity**
  - A constant the fixes the number of arguments

Syntax of FOL: Basic elements

- Constants KingJohn, 2, NUS,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
- Equality =
- Quantifiers $\forall$, $\exists$

First-order logic semantics

- Semantics relates sentences to models in order to determine truth
  - Interpretation accomplishes semantics: it maps symbols to models
    - constant symbols → objects
    - predicate symbols → relations
    - function symbols → functions

Term

- A logical expression that refers to an object
  - Constants
    - We could assign names to all objects, like providing a name for every shoe in your closet
    - It is not always convenient to have a distinct symbol to name every objects. (Use function)
  - Function symbols
    - Replaces the need to name all the shoes
    - OnLeftFoot(John))
      - Refers to a shoe, some shoe
  - A complex term is just a complicated kind of name.
  - Semantic of sentence: f(t1, …, tn)

Atomic Sentences

- Formed by a predicate symbol followed by parenthesized list of terms
  - Sibling (Alice, Bob)
  - Married (Father(Alice), Mother(Bob))

- **An atomic sentence is true in a given model, under a given interpretation, if the relation referred to by the predicate symbol holds among the objects referred to by the arguments**

Complex sentences

- We can use logical connectives to construct more complex sentences.
  - ~Sibling(LeftLeg(Alice), Bob)
  - Sibling(Alice, Bob) ^ Sibling (Bob, Alice)

$$\neg S,\ S_1 \wedge S_2,\ S_1 \vee S_2,\ S_1 \Rightarrow S_2,\ S_1 \Leftrightarrow S_2,$$

E.g. *Sibling(KingJohn,Richard)* $\Rightarrow$ *Sibling(Richard,KingJohn)*

Universal quantification

- ***∀<variables> <sentence>***
-

Everyone at NUS is smart:

$\forall x\ At(x,NUS) \Rightarrow Smart(x)$

$\forall x\ P$ is true in a model *m* iff *P* is true with *x* being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of *P*

$$
\begin{aligned}
& At(KingJohn,NUS) \Rightarrow Smart(KingJohn) \\
\wedge\ & At(Richard,NUS) \Rightarrow Smart(Richard) \\
\wedge\ & At(NUS,NUS) \Rightarrow Smart(NUS) \\
\wedge\ & \ldots
\end{aligned}
$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
-
- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall$x At(x,NUS) $\wedge$ Smart(x)

  means “Everyone is at NUS and everyone is smart”

Existential quantification

- **$\exists$<*variables*> <*sentence*>**

- Someone at NUS is smart:
- $\exists x$ At(x,NUS) $\wedge$ Smart(x)
-
- $\exists x$ *P* is true in a model *m* iff *P* is true with *x* being some possible object in the model
-
- Roughly speaking, equivalent to the disjunction of instantiations of *P*
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At(KingJohn,NUS) $\wedge$ Smart(KingJohn)
$\vee$ At(Richard,NUS) $\wedge$ Smart(Richard)
$\vee$ At(NUS,NUS) $\wedge$ Smart(NUS)
$\vee$ ...

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:
-

$$\exists x \text{ At(x,NUS)} \Rightarrow \text{Smart(x)}$$

is true if there is anyone who is not at NUS!

Properties of quantifiers

- $\forall x \, \forall y$ is the same as $\forall y \, \forall x$
- $\exists x \, \exists y$ is the same as $\exists y \, \exists x$
- $\exists x \, \forall y$ is not the same as $\forall y \, \exists x$
- $\exists x \, \forall y$ Loves(x,y)
  - "There is a person who loves everyone in the world"
  -
- $\forall y \, \exists x$ Loves(x,y)
  - "Everyone in the world is loved by at least one person"
  -
- Quantifier duality: each can be expressed using the other
-
- $\forall x$ Likes(x,IceCream) $\neg\exists x \, \neg$Likes(x,IceCream)
-
- $\exists x$ Likes(x,Broccoli) $\neg\forall x \, \neg$Likes(x,Broccoli)
-

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
-
- E.g., definition of *Sibling* in terms of *Parent*:
-

$$\forall x,y\ Sibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg (m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$$

Using FOL

Using FOL

- **Assertion**: Sentences are added to KB base using TELL.
  - TELL (KB, King(John)).
  - TELL(KB, $\forall$x King(x) $\Rightarrow$Person(x)).
- **Queries or Goals**: Questions asked using ASK.
  - ASK(KB, Person(John)), ASK(KB, $\exists$x Person(x))
- **Answer**: The standard form for an answer is a substitution or binding list, which is a set of **variable/term** pairs. {x/John}

The kinship domain:

- Brothers are siblings
- $\forall$x,y *Brother(x,y)* $\Leftrightarrow$ *Sibling(x,y)*
-
- One's mother is one's female parent
- $\forall$m,c *Mother(c)* = m $\Leftrightarrow$ *(Female(m)* $\wedge$ *Parent(m,c))*
-
- “Sibling” is symmetric
- $\forall$x,y *Sibling(x,y)* $\Leftrightarrow$ *Sibling(y,x)*
-

Example sentences

- Brothers are siblings
  - $\forall$ x, y Brother(x, y) $\Rightarrow$ Sibling(x, y)
- Sibling is transitive
  - $\forall$ x, y, z Sibling(x, y) $\wedge$ Sibling(y, z) $\Rightarrow$ Sibling(x, z)
- One's mother is one's sibling's mother
  - $\forall$ m, c Mother(m, c) $\wedge$ Sibling(c, d) $\Rightarrow$ Mother(m, d)
- A first cousin is a child of a parent's sibling
  - $\forall$ c, d FirstCousin(c, d) $\Leftrightarrow$ $\exists$ p, ps Parent(p, d) $\wedge$ Sibling(p, ps) $\wedge$Parent(ps, c)

Using FOL

The set domain:

Symbol:{}, x, Set, $\subseteq$, $\cap$, $\cup$, $\in$…

$\forall s\ Set(s) \Leftrightarrow (s = \{\}) \lor (\exists x,s_2\ Set(s_2) \land s = \{x|s_2\})$

$\neg\exists x,s\ \{x|s\} = \{\}$

$\forall x,s\ x \in s \Leftrightarrow s = \{x|s\}$

$\forall x,s\ x \in s \Leftrightarrow [\ \exists y,s_2\}\ (s = \{y|s_2\} \land (x = y \lor x \in s_2))]$

$\forall s_1,s_2\ s_1 \subseteq s_2 \Leftrightarrow (\forall x\ x \in s_1 \Rightarrow x \in s_2)$

$\forall s_1,s_2\ (s_1 = s_2) \Leftrightarrow (s_1 \subseteq s_2 \land s_2 \subseteq s_1)$

$\forall x,s_1,s_2\ x \in (s_1 \cap s_2) \Leftrightarrow (x \in s_1 \land x \in s_2)$

$\forall x,s_1,s_2\ x \in (s_1 \cup s_2) \Leftrightarrow (x \in s_1 \lor x \in s_2)$